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A Class of One-Step Hybrid Third Derivative Block Method for the Direct Solution of Initial Value Problems of Second-Order Ordinary Differential Equations

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Abstract

In this paper, we consider the development of a class of one-step hybrid third derivative block method with three off-grid points for the direct solution of initial value problems of second order Ordinary Differential Equations. We adopted method of interpolation and collocation of power series approximate solution to generate the continuous hybrid linear multistep method, which was evaluated at grid points to give a continuous block method. The discrete block method was recovered when the continuous block method was evaluated at selected grid points. The basic properties of the method was investigated and was found to be zero-stable, consistent and convergent. The efficiency of the method was tested on some stiff equations and was found to give better approximation than the existing method, which we compared our result with.

Keywords

One-Step; Hybrid Block Method; Third Derivative; Stiff Odes; Collocation and Interpolation Method.

Introduction

This paper solves second order initial value problems in the form

$$y'' = f\left(x, y(x), y'(x)\right), \ y(x_0) = y_0, \ y'(x_0) = y'_0 \tag{1}$$

where f is continuous within the interval of integration. Solving higher order derivatives method by reducing them to a system of first-order approach involves more functions to evaluate which then leads to a computational burden as in [1-3]. The setbacks of this approach had been reported by scholars, among them are Bun and Vasil'yer and Awoyemi et al [4,5].

The method of collocation and interpolation of the power series approximation to generate continuous linear multistep method has been adopted by many scholars; among them are Fatunla, Awoyemi, Olabode, Vigor Aquilar and Ramos, Adeniran et al, Abdelrahim et al, Mohammad et al to mention a few [5-11]. Block method generates independent solution at selected grid points without overlapping. It is less expensive in terms of number of function evaluation compared to predictor corrector method, moreover it possess the properties of Runge Kutta method for being self-starting and does not require starting values. Some of the authors that proposed block method are: [12-17].

In this paper, we developed a on e-step hybrid third derivative method with three offgrid, which is implemented in block method. The method is self-starting and does not require starting values or predictors. The implementation of the method is cheaper than the predictor-corrector method. This method harnesses the properties of hybrid and third derivative, this makes it e¢cient for sti¤ problems.

The paper is organized as follows: In section 2, we discuss the methods and the materials for the development of the method. Section 3 considers analysis of the basis properties of the method which include convergence and stability region, numerical experiments where the e¢ciency of the derived method is tested on some numerical examples and discussion of results. Lastly, we concluded in section 4.

Derivation of the Method

We consider a power series approximate solution of the form

$$y(x) = \sum_{j=0}^{2r+s-1} a_i \left(\frac{x - x_n}{h}\right)^j \tag{2}$$

where r=2 and s=5 are the numbers of interpolation and collocation points respectively, is considered t o be a solution to (1).

The second and third derivative of (2) gives

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$$y''(x) = \sum_{j=2}^{2r+s-1} \frac{a_j j!}{h^2 (j-2)} \left(\frac{x-x_n}{h}\right)^{j-2}$$
$$= f(x, y, y'),$$

$$y'''(x) = \sum_{j=3}^{2r+s-1} \frac{a_j j!}{h^3 (j-3)} \left(\frac{x-x_n}{h}\right)^{j-3}$$
$$= g(x, y, y'), \tag{3}$$

Substituting (3) into (1) gives

$$f(x, y, y'') = \sum_{j=2}^{2r+s-1} \frac{a_j j!}{h^2 (j-2)} \left(\frac{x-x_n}{h}\right)^{j-2} + \sum_{j=3}^{2r+s-1} \frac{a_j j!}{h^3 (j-3)} \left(\frac{x-x_n}{h}\right)^{j-3} \tag{4}$$

Collocating (4) at all points $x_{n+r}, r = 0\left(\frac{1}{4}\right)1$ and Interpolating Equation (2) at $x_{n+s}, s = 0, \frac{1}{4}$, gives a system of non linear equation of the form

$$AX = U \tag{5}$$

Where

$$A = [a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}]^T,$$

$$U = \left[y_n, y_{n+\frac{1}{4}}, f_n, f_{n+\frac{1}{4}}, f_{n+\frac{1}{2}}, f_{n+\frac{3}{4}}, f_{n+1}, g_n, g_{n+\frac{1}{4}}, g_{n+\frac{1}{2}}, g_{n+\frac{3}{4}}, g_{n+1}\right]^T,$$

and

[matrix equation]

Solving (5) for $a_i's$ using Gaussian elimination method, gives a continuous hybrid linear multistep method of the form

$$y(x) = \sum_{j=0,\frac{1}{4}} \alpha_j y_{n+j} + h^2\left[\sum_{j=0}^{1} \beta_j f_{n+j} + \beta f_{n+}\right] + h^3\left[\sum_{j=0}^{1} \gamma_j g_{n+j} + \gamma g_{n+}\right], k = \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \tag{6}$$

Differentiating (6) once yields

$$p'(x) = \frac{1}{h} \sum_{j=0,\frac{1}{4}} \alpha_j y_{n+j} + h\left[\sum_{j=\frac{1}{4},\frac{1}{2},\frac{3}{4}} \beta f_{n+} + \sum_{j=0}^{1} \beta f_{n+}\right] + h^2\left[\sum_{j=\frac{1}{4},\frac{1}{2},\frac{3}{4}} \gamma g_{n+} + \sum_{j=0}^{1} \gamma g_{n+}\right] \tag{7}$$

Where

$$\alpha_0 = 1 - \frac{4(x-x_n)}{h}$$

$$\alpha_{\frac{1}{4}} = \frac{4(x-x_n)}{h}$$

Equation (6) is evaluated at the non-interpolating points and (7) at all points, produces the following general equations in block form

$$\beta_0 = -\frac{2602339}{38320128}(x-x_n)h + \frac{1}{2}(x-x_n)^2 - \frac{485}{36}\frac{(x-x_n)^4}{h^2}$$

$$+\frac{4031}{54}\frac{(x-x_n)^5}{h^3} - \frac{85862}{405}\frac{(x-x_n)^6}{h^4}$$

$$+\frac{208100}{567}\frac{(x-x_n)^7}{h^5} - \frac{75920}{189}\frac{(x-x_n)^8}{h^6} + \frac{66080}{243}\frac{(x-x_n)^9}{h^7}$$

$$-\frac{126464}{1215}\frac{(x-x_n)^{10}}{h^8} + \frac{5120}{297}\frac{(x-x_n)^{11}}{h^9}$$

$$\beta_{\frac{1}{4}} = -\frac{148231}{11975040}(x-x_n)h - \frac{128}{9}\frac{(x-x_n)^4}{h^2}$$

$$+\frac{7168}{45}\frac{(x-x_n)^5}{h^3} - \frac{270592}{405}\frac{(x-x_n)^6}{h^4}$$

$$+\frac{841216}{567}\frac{(x-x_n)^7}{h^5}$$

$$-\frac{364288}{189}\frac{(x-x_n)^8}{h^6} + \frac{358912}{243}\frac{(x-x_n)^9}{h^7}$$

$$-\frac{753664}{1215}\frac{(x-x_n)^{10}}{h^8} + \frac{32768}{297}\frac{(x-x_n)^{11}}{h^9}$$

$$\beta_{\frac{1}{2}} = -\frac{1807}{80640}(x-x_n)h - \frac{12(x-x_n)^4}{h^2}$$

$$-\frac{456}{5}\frac{(x-x_n)^5}{h^3} - \frac{4424}{15}\frac{(x-x_n)^6}{h^4} - \frac{10496}{21}\frac{(x-x_n)^7}{h^5}$$

$$+\frac{3264}{7}\frac{(x-x_n)^8}{h^6} - \frac{2048}{9}\frac{(x-x_n)^9}{h^7} + \frac{2048}{45}\frac{(x-x_n)^{10}}{h^8}$$

$$\beta_{\frac{3}{4}} = -\frac{243193}{11975040}(x-x_n)h + \frac{128}{9}\frac{(x-x_n)^4}{h^2}$$

$$-\frac{17408}{135}\frac{(x-x_n)^5}{h^3} + \frac{213248}{405}\frac{(x-x_n)^6}{h^4}$$

$$-\frac{685568}{567}\frac{(x-x_n)^7}{h^5} + \frac{313088}{189}\frac{(x-x_n)^8}{h^6}$$

$$-\frac{326144}{243}\frac{(x-x_n)^9}{h^7} + \frac{720896}{1215}\frac{(x-x_n)^{10}}{h^8}$$

$$-\frac{32768}{297}\frac{(x-x_n)^{11}}{h^9}$$

$$\beta_1 = -\frac{382169}{191600640}(x-x_n)h + \frac{53}{36}\frac{(x-x_n)^4}{h^2}$$

$$-\frac{1241}{90}\frac{(x-x_n)^5}{h^3} + \frac{23758}{405}\frac{(x-x_n)^6}{h^4}$$

$$-\frac{80356}{567}\frac{(x-x_n)^7}{h^5} + \frac{38992}{189}\frac{(x-x_n)^8}{h^6}$$

$$-\frac{43552}{243}\frac{(x-x_n)^9}{h^7} + \frac{103936}{1215}\frac{(x-x_n)^{10}}{h^8}$$

$$-\frac{5120}{297}\frac{(x-x_n)^{11}}{h^9}$$

$$\gamma_0 = -\frac{28343}{18247680}(x-x_n)h^2 + \frac{1}{6}(x-x_n)^3$$

$$-\frac{25}{18}\frac{(x-x_n)^4}{h} + \frac{209}{36}\frac{(x-x_n)^5}{h^2} - \frac{398}{27}\frac{(x-x_n)^6}{h^3}$$

$$+\frac{4546}{189}\frac{(x-x_n)^7}{h^4} - \frac{1600}{63}\frac{(x-x_n)^8}{h^5} + \frac{1360}{81}\frac{(x-x_n)^9}{h^6}$$

$$-\frac{512}{81}\frac{(x-x_n)^{10}}{h^7} + \frac{512}{495}\frac{(x-x_n)^{11}}{h^8}$$

$$\gamma_{\frac{1}{4}} = \frac{551}{49896}(x-x_n)h^2 - \frac{16}{3}\frac{(x-x_n)^4}{h}$$

$$+\frac{608}{15}\frac{(x-x_n)^5}{h^2} - \frac{18848}{135}\frac{(x-x_n)^6}{h^3} + \frac{7424}{27}\frac{(x-x_n)^7}{h^4}$$

$$-\frac{20768}{63}\frac{(x-x_n)^8}{h^5} + \frac{19328}{81}\frac{(x-x_n)^9}{h^6} - \frac{38912}{405}\frac{(x-x_n)^{10}}{h^7}$$

$$+\frac{8192}{495}\frac{(x-x_n)^{11}}{h^8}$$

$$\gamma_{\frac{1}{2}} = \frac{32027}{3548160}(x-x_n)h^2 - \frac{6(x-x_n)^4}{h}$$

$$+\frac{264}{5}\frac{(x-x_n)^5}{h^2} - \frac{3124}{15}\frac{(x-x_n)^6}{h^3} + \frac{3224}{7}\frac{(x-x_n)^7}{h^4}$$

$$-\frac{608(x-x_n)^8}{h^5} + \frac{4288}{9}\frac{(x-x_n)^9}{h^6} - \frac{1024}{5}\frac{(x-x_n)^{10}}{h^7}$$

$$+\frac{2048}{55}\frac{(x-x_n)^{11}}{h^8}$$

$$\gamma_{\frac{3}{4}} = \frac{3959}{1596672}(x-x_n)h^2 - \frac{16}{9}\frac{(x-x_n)^4}{h}$$

$$+\frac{736}{45}\frac{(x-x_n)^5}{h^2} - \frac{9184}{135}\frac{(x-x_n)^6}{h^3} + \frac{30208}{189}\frac{(x-x_n)^7}{h^4}$$

$$-\frac{14176}{63}\frac{(x-x_n)^8}{h^5} + \frac{15232}{81}\frac{(x-x_n)^9}{h^6} - \frac{34816}{405}\frac{(x-x_n)^{10}}{h^7}$$

$$+\frac{8192}{495}\frac{(x-x_n)^{11}}{h^8}$$

$$\gamma_1 = \frac{14339}{127733760}(x-x_n)h^2 - \frac{1}{12}\frac{(x-x_n)^4}{h}$$

$$+\frac{47}{60}\frac{(x-x_n)^5}{h^2} - \frac{452}{135}\frac{(x-x_n)^6}{h^3} + \frac{1538}{189}\frac{(x-x_n)^7}{h^4}$$

$$-\frac{752}{63}\frac{(x-x_n)^8}{h^5} + \frac{848}{81}\frac{(x-x_n)^9}{h^6} - \frac{2048}{405}\frac{(x-x_n)^{10}}{h^7}$$

$$+\frac{512}{495}\frac{(x-x_n)^{11}}{h^8}$$

Equation (6) is evaluated at the non-interpolating points

$$\left\{x_{n+\frac{1}{2}}, x_{n+\frac{3}{4}}, x_{n+1}\right\} \quad \text{and (7) at all points}$$

$$\left\{x_n, x_{n+\frac{1}{4}}, x_{n+\frac{1}{2}}, x_{n+\frac{3}{4}}, x_{n+1}\right\}$$

produces the following general equations in block form

$$A Y_L = B R_1 + C R_2 + D R_3 + E R_4 + G R_5 \tag{8}$$

Where,

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{4}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{4}{h} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{4}{h} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{4}{h} & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\frac{4}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 0 \\ \frac{4}{h} & 1 \\ \frac{4}{h} & 0 \\ \frac{4}{h} & 0 \\ \frac{4}{h} & 0 \\ \frac{4}{h} & 0 \end{bmatrix}, R_1 = \begin{bmatrix} y_n \\ y'_n \end{bmatrix}$$

$$Y = \begin{bmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{4}} \\ y_{n+1} \\ y'_{n+\frac{1}{4}} \\ y'_{n+\frac{1}{2}} \\ y'_{n+\frac{3}{4}} \\ y'_{n+1} \end{bmatrix} \quad C = \begin{bmatrix} \frac{37111h^2}{6967296} \\ \frac{12679h^2}{1161216} \\ \frac{19709h^2}{1161216} \\ -\frac{260233h}{38320128} \\ \frac{1961683h}{95800320} \\ \frac{1403419h}{63866880} \\ \frac{2182739h}{95800320} \\ \frac{5021969h}{191600640} \end{bmatrix}, R_2 = [f_n]$$

$$D = \begin{bmatrix} \frac{16463h^2}{435456} & \frac{445h^2}{32256} & \frac{2225h^2}{435456} & \frac{3217h^2}{6967296} \\ \frac{6149h^2}{72576} & \frac{767h^2}{10752} & \frac{1403h^2}{72576} & \frac{1381h^2}{1161216} \\ \frac{9925h^2}{72576} & \frac{767h^2}{5376} & \frac{5179h^2}{72576} & \frac{8411h^2}{1161216} \\ \frac{-148231h}{11975040} & \frac{-1807h}{80640} & \frac{-243193h}{11975040} & \frac{-382169h}{191600640} \\ \frac{83485h}{1197504} & \frac{1489h}{80640} & \frac{89279h}{5987520} & \frac{137161h}{95800320} \\ \frac{26921h}{147840} & \frac{767h}{5376} & \frac{103853h}{3991680} & \frac{5303h}{2365440} \\ \frac{1156801h}{5987520} & \frac{21521h}{80640} & \frac{165731h}{1197504} & \frac{358217h}{95800320} \\ \frac{2735353h}{11975040} & \frac{24817h}{80640} & \frac{2640391h}{11975040} & \frac{3530299h}{38320128} \end{bmatrix}, R_3 = \begin{bmatrix} f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{3}{4}} \\ f_{n+1} \end{bmatrix}$$

$$
G = \begin{bmatrix}
\dfrac{-17h^3}{9072} & \dfrac{-83h^3}{30720} & \dfrac{-871h^3}{1451520} & \dfrac{-599h^3}{23224320} \\
\dfrac{-3053h^3}{967680} & \dfrac{-83h^3}{15360} & \dfrac{-347h^3}{193536} & \dfrac{-127h^3}{1935360} \\
\dfrac{-29h^3}{7560} & \dfrac{-83h^3}{15360} & \dfrac{-269h^3}{241920} & \dfrac{-1117h^3}{3870720} \\
\dfrac{551h^2}{49896} & \dfrac{32027h^2}{3548160} & \dfrac{3959h^2}{1596672} & \dfrac{14339h^2}{127733760} \\
\dfrac{-23851h^2}{2280960} & \dfrac{-12053h^2}{1774080} & \dfrac{-5755h^2}{3193344} & \dfrac{-2567h^2}{31933440} \\
\dfrac{-95h^2}{16632} & \dfrac{-56677h^2}{3548160} & \dfrac{-7951h^2}{2661120} & \dfrac{-5311h^2}{42577920} \\
\dfrac{-72269h^2}{15966720} & \dfrac{-12053h^2}{1774080} & \dfrac{-123463h^2}{15966720} & \dfrac{-6439h^2}{31933440} \\
\dfrac{-61h^2}{249480} & \dfrac{32027h^2}{3548160} & \dfrac{15701h^2}{1140480} & \dfrac{-312317h^2}{127733760}
\end{bmatrix}, \quad
R_5 = \begin{bmatrix} g_{n+\frac{1}{4}} \\ g_{n+\frac{1}{2}} \\ g_{n+\frac{3}{4}} \\ g_{n+1} \end{bmatrix}
$$

$$
E = \begin{bmatrix}
\dfrac{4253h^3}{23224320} \\
\dfrac{23h^3}{60480} \\
\dfrac{467h^3}{774144} \\
\dfrac{-28343h^2}{18247680} \\
\dfrac{551h^2}{798336} \\
\dfrac{3629h^2}{4730880} \\
\dfrac{3239h^2}{3991680} \\
\dfrac{25651h^2}{25546752}
\end{bmatrix}, \quad
R_4 = [g_n]
$$

Multiplying equation (8) by the inverse of (A) gives the hybrid block formula of the form

$$ I\,Y_L = \overline{B}\,R_1 + \overline{C}\,R_2 + \overline{D}\,R_3 + \overline{E}\,R_4 + \overline{G}\,R_5 $$

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{4}} \\ y_{n+1} \\ y'_{n+\frac{1}{4}} \\ y'_{n+\frac{1}{2}} \\ y'_{n+\frac{3}{4}} \\ y'_{n+1}
\end{bmatrix}
=
\begin{bmatrix}
1 & \frac{h}{4} \\
1 & \frac{h}{2} \\
1 & \frac{3h}{4} \\
1 & h \\
0 & 1 \\
0 & 1 \\
0 & 1 \\
0 & 1
\end{bmatrix}
\begin{bmatrix} y_n \\ y'_n \end{bmatrix}
+
\begin{bmatrix}
\dfrac{2602339h^2}{153280512} \\
\dfrac{35h^2}{891} \\
\dfrac{39015h^2}{630784} \\
\dfrac{6353h^2}{74844} \\
\dfrac{1539551h}{17418240} \\
\dfrac{24463h}{272160} \\
\dfrac{6501h}{71680} \\
\dfrac{1601h}{17010}
\end{bmatrix}
[f_n]
$$

$$
+
\begin{bmatrix}
\dfrac{148231h^2}{47900160} & \dfrac{1807h^2}{322560} & \dfrac{243193h^2}{47900160} & \dfrac{382169h^2}{766402560} \\
\dfrac{196h^2}{4455} & \dfrac{40h^2}{63} & \dfrac{68h^2}{4455} & \dfrac{13h^2}{8910} \\
\dfrac{18531h^2}{197120} & \dfrac{3159h^2}{35840} & \dfrac{6813h^2}{197120} & \dfrac{8469h^2}{3153920} \\
\dfrac{13952h^2}{93555} & \dfrac{52h^2}{315} & \dfrac{8578h^2}{93555} & \dfrac{3457h^2}{374220} \\
\dfrac{89371h}{1088640} & \dfrac{103h}{2520} & \dfrac{38341h}{1088640} & \dfrac{59681h}{17418240} \\
\dfrac{1654h}{8505} & \dfrac{52h}{315} & \dfrac{394h}{8505} & \dfrac{1153h}{272160} \\
\dfrac{921h}{4480} & \dfrac{81h}{280} & \dfrac{711h}{4480} & \dfrac{411h}{71680} \\
\dfrac{2048h}{8505} & \dfrac{104h}{315} & \dfrac{2048h}{8505} & \dfrac{1601h}{17010}
\end{bmatrix}
\begin{bmatrix} f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{3}{4}} \\ f_{n+1} \end{bmatrix}
+
\begin{bmatrix}
\dfrac{28343h^3}{72990720} \\
\dfrac{1277h^3}{1330560} \\
\dfrac{9747h^3}{6307840} \\
\dfrac{269h^2}{124740} \\
\dfrac{26051h^2}{11612160} \\
\dfrac{421h^2}{181440} \\
\dfrac{339h^2}{143360} \\
\dfrac{29h^2}{11340}
\end{bmatrix}
[g_n]
+
\begin{bmatrix}
\dfrac{-551h^3}{199584} & \dfrac{-32027h^3}{14192640} & \dfrac{-3959h^3}{6386688} & \dfrac{-14339h^2}{510935040} \\
\dfrac{-41h^3}{5544} & \dfrac{-5h^3}{693} & \dfrac{-17h^3}{9240} & \dfrac{-109h^2}{1330560} \\
\dfrac{-4509h^3}{394240} & \dfrac{-19197h^3}{1576960} & \dfrac{-9h^3}{2464} & \dfrac{-27h^2}{180224} \\
\dfrac{-464h^3}{31185} & \dfrac{-10h^3}{693} & \dfrac{-16h^3}{4455} & \dfrac{-5h^2}{12474} \\
\dfrac{-31207h^2}{1451520} & \dfrac{-81h^2}{1520} & \dfrac{-1243h^2}{290304} & \dfrac{-2237h^2}{11612160} \\
\dfrac{-19h^2}{1134} & \dfrac{-h^2}{40} & \dfrac{-31h^2}{5670} & \dfrac{-43h^2}{181440} \\
\dfrac{-279h^2}{17920} & \dfrac{-81h^2}{1520} & \dfrac{-183h^2}{17920} & \dfrac{-9h^2}{28672} \\
\dfrac{-32h^2}{2835} & 0 & \dfrac{32h^2}{2835} & \dfrac{-29h^2}{11340}
\end{bmatrix}
\begin{bmatrix} g_{n+\frac{1}{4}} \\ g_{n+\frac{1}{2}} \\ g_{n+\frac{3}{4}} \\ g_{n+1} \end{bmatrix}
$$

Analysis of Basic Properties of the Method
Order of the Block

According to [7] the order of the new method in Equation (8) is obtained by using the Taylor series and it is found that the developed method has a uniformly order eleven, with an error constants vector of:

$$C_{11} = \begin{bmatrix} 6.1829 \times 10^{-14}, 1.752 \times 10^{-13}, 3.0454 \times 10^{-13}, & 4.8542 \times 10^{-13}, \\ 4.1791 \times 10^{-13}, 4.8542 \times 10^{-13}, 5.5292 \times 10^{-13}, 9.7083 \times 10^{-13} \end{bmatrix}^{T}$$

Consistency

The hybrid block method [4] is said to be consistent if it has an order more than or equal to one.

Therefore, our method is consistent.

Zero Stability of Our Method

Definition: A block method is said to be zero-stable if as $h \to 0$, the root $z_i, i = 1(1)k$ of the first characteristic polynomial $\rho(z) = 0$ that is

$$\rho(z) = \det\left[\sum_{j=0}^{k} A^{(i)} z^{k-i} \right] = 0$$

Satisfies $|z_i| \le 1$ and for those roots with $|z_i| = 1$, multiplicity must not exceed two. The block method for k=1, with three off grid collocation point expressed in the form

$$\rho(z) = z \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{h}{4} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{h}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{3h}{4} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & h \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = z^6(z-1)^2$$

$$\rho(z) = z^6(z-1)^2 = 0,$$

Hence, our method is zero-stable.

Regions of Absolute Stability (RAS)

Using Mat Lab package, we were able to plot the stability region of the block methods (see figure 1 below).

The stability polynomial for K=1 with three offstep point using Scientific Workplace software package we have in the following one:

$$= h^{12}\left(\left(\frac{7219}{15107327262720000}\right)w^4 - \left(\frac{79}{156959244288000}\right)w^3\right) - h^{11}\left(\left(\frac{3423139}{139591703907532800}\right)w^4 + \left(\frac{1171}{30519853056000}\right)w^3\right)$$

$$- h^{10}\left(\left(\frac{4666483}{3625758543052800}\right)w^3 + \left(\frac{585349}{23930006384148480}\right)w^4\right) + h^{9}\left(\left(\frac{73634299}{2913555972096000}\right)w^3 - \left(\frac{2899283}{1495625399009280}\right)w^3\right)$$

$$- h^{8}\left(\left(\frac{12604072069}{1495625399000928000}\right)w^4 + \left(\frac{6897035147}{8157956721868800}\right)w^3\right) - h^{7}\left(\left(\frac{775601}{9442079539200}\right)w^4 + \left(\frac{707100649}{109258348953600}\right)w^3\right)$$

$$+ h^{6}\left(\left(\frac{1953881}{256927334400}\right)w^4 - \left(\frac{26693444639}{145677798604800}\right)w^3\right) - h^{5}\left(\left(\frac{2850443}{36883123200}\right)w^4 + \left(\frac{357859}{1077753600}\right)w^3\right)$$

$$- h^{4}\left(\left(\frac{2587}{4257792}\right)w^4 + \left(\frac{12221035}{1034643456}\right)w^3\right) + h^{3}\left(\left(\frac{25}{1782}\right)w^4 + \left(\frac{269}{114740}\right)w^3\right) - h^{2}\left(\left(\frac{16399}{228096}\right)w^4 + \left(\frac{1009733}{4790016}\right)w^3\right) + w^4 - w^3$$

x-values	Exact Solution	Computed Solution	Error in our method	Error in [13]
0.100	0.90483741803595957316	0.90483741803595957245	7.100E(-19)	1.054712E(-14)
0.200	0.81873075307798185867	0.81873075307798185795	7.200E(-19)	1.776357E(-14)
0.300	0.74081822068171786607	0.74081822068171786534	7.300E(-19)	2.342571E(-14)
0.400	0.67032004603563930074	0.67032004603563930001	7.300E(-19)	2.797762E(-14)
0.500	0.60653065971263342360	0.60653065971263342285	7.500E(-19)	3.130829E(-14)
0.600	0.54881163609402643263	0.54881163609402643185	7.800E(-19)	3.397282E(-14)
0.700	0.49658530379140951470	0.49658530379140951390	8.000E(-19)	3.563816E(-14)
0.800	0.44932896411722159143	0.44932896411722159058	8.500E(-19)	3.674838E(-14)
0.900	0.40656965974059911188	0.40656965974059911099	8.900E(-19)	3.730349E(-14)
1.00	0.36787944117144232160	0.36787944117144232065	9.500E(-19)	3.741452E(-14)

Table 1: Comparison of the proposed method

| x-values	Exact Solution	Computed Solution	Error in our method	Error in [16]
.0.100	-0.10517091807564762480	-0.10517091807564762481	1.000E(-20)	8.326679(-17)
0.200	-0.22140275816016983390	-0.22140275816016983391	1.000E(-20)	2.775557(-16)
0.300	-0.34985880757600310400	-0.34985880757600310397	-3.000E(-20)	5.551115(-16)
0.400	-0.49182469764127031780	-0.49182469764127031779	-1.000E(-20)	9.436896(-16)
0.500	-0.64872127070012814680	-0.64872127070012814679	-1.000E(-20)	2.109424(-15)
0.600	-0.82211880039050897490	-0.82211880039050897478	-1.000E(-19)	3.219647(-15)
0.700	-1.01375270747047652160	-1.01375270747047652150	-1.000E(-19)	4.440892(-15)
0.800	-1.22554092849246760460	-1.22554092849246760440	-2.000E(-19)	5.995204(-15)
0.900	-1.45960311115694966380	-1.45960311115694966350	-3.000E(-19)	7.771561(-15)
1.00	-1.71828182845904523540	-1.71828182845904523500	-4.000E(-19)	1.065814(-14)

Table 2
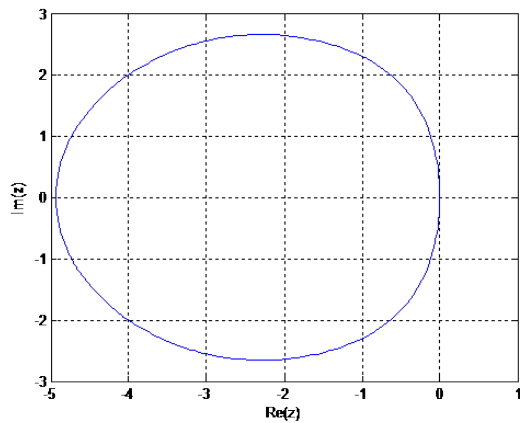
Figure 1: Absolute Stability Region

Using Mat Lab software, the absolute stability region of the new method is plotted and shown in figure 1.

Numerical Example

Problem I: We consider a highly stiff problem

$$y'' + 1001y' + 1000y, \quad y(0) = 1, y'(0) = -1$$

Exact Solution: $y(x) = \exp(-x)\ h = \dfrac{1}{10}$

Problem II:

$$f(x, y, y') = y', \quad y(0) = 1, y'(0) = \frac{1}{2}, \ 0 \le x \le 1.$$

Exact Solution: $y(x) = 1 - e^x\ \ with\ h = \dfrac{1}{100}$

Conclusion

It is evident from the above tables that our proposed methods are indeed accurate, and can handle stiff equations. Also in terms of stability analysis, the method is A-stable.

Comparing the new method with the existing method, the result presented in the tables 1 and 2 shows that the new method performs better than the existing method, and even the order of new method is higher than the order of the existing method [11,18]. In this article, a one-step block method with three off-step points is derived via the interpolation and collocation approach. The developed method is consistent, A-stable, convergent, with a region of absolute stability and order Ten.

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